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On certain two-dimensional conservative mechanical systems with a cubic second integral

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Abstract

In a previous paper (Yehia H M 1986 *J. Mec. Theor. Appl.* **5** 55–71) we have introduced a method for constructing integrable conservative two-dimensional mechanical systems whose second integral of motion is polynomial in the velocities. This method has proved successful in constructing a multitude of irreversible systems (involving gyroscopic forces) with a second quadratic integral (Yehia H M 1992 *J. Phys. A: Math. Gen.* **25** 197–221). The objective of this paper is to apply the same method for the systematic construction of mechanical systems with a cubic integral. As in our previous works, the configuration space is not assumed to be a Euclidean plane. This widens the range of applicability of the results to diverse mechanical systems to include such problems as rigid body dynamics. Several new reversible and irreversible integrable systems are obtained. Some of these systems generalize previously known ones by introducing additional parameters which may change either or both of the configuration manifold and the potential of the forces acting on the system. Other systems are completely new. An application is given to problems of rigid body dynamics. The famous classical integrable case due to Goriachev and Chaplygin and all its subsequent generalizations by several authors are further generalized to include certain variable gyroscopic forces that preserve a cubic integral. On the other hand, the above case and another less famous case of rigid body dynamics due to Goriachev and the Hall–Toda case of particle mechanics are all obtained as special cases, corresponding to different choices of certain parameters, from one more general system unifying them all.

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1. Introduction

It is well known that mechanical systems are, as a rule, nonintegrable. Nevertheless, there is no general method for deciding about the integrability of a given mechanical system and, in the case of integrability, to construct its integrals. It is thus of great importance to construct new integrable systems and to tabulate all the known ones of certain common types. Enormous efforts have been dedicated to this direction of research over the last few decades. Most of the interest has concentrated on the system composed of a single particle moving in the plane under the action of stationary potential forces whose second integral is polynomial in the velocities. The problem for this system is to find the potential compatible with a given ansatz of the integral and, in the process, to determine the coefficients in this integral. However, only a few integrable cases have been found. In the majority of these cases, the additional first integral is a polynomial in the generalized velocities. For a nearly complete review up to 1985 see [3]. The diversity of methods for obtaining the additional integral in each of these cases points to the need for a unified and systematic criterion for their detection.

The above problem can be generalized in two different directions to enhance the applicability of the results, while preserving the frame of investigation in two dimensions. Firstly, we do not restrict the configuration space to the Euclidean plane. Secondly, we allow gyroscopic forces (conservative zero-potential forces), that is forces appearing due to terms linear in the velocities in the Lagrangian of the system (see, for example, [2]). In this new setting, the above problem includes a wide class of mechanical problems. To this type belongs, for example, the problem of motion of a natural mechanical system with n degrees of freedom, having $n-2$ cyclic coordinates. Another example is the problem of motion of a particle on a smooth (fixed or rotating) surface under a variety of forces. Further examples are given by the problem of motion about a fixed point of a rigid body acted upon by potential and gyroscopic forces that allow a cyclic variable [5, 6]. The general problem of inertial motion of a body in an infinite ideal incompressible fluid has been shown to belong to the type under consideration [8].

All such systems can be described by or reduced to a system with Lagrangian

$$L = \frac{1}{2}\left(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2\right) + a_1\dot{q}_1 + a_2\dot{q}_2 - V \tag{1}$$

where the six coefficients a_{ij}, a_i, V are functions of q_1, q_2 and the dots denote differentiation with respect to time t. According to a theorem of Birkhoff [9], there always exists a coordinate transformation[1] that reduces equation (1) to a system of the type

$$L = \frac{1}{2}\Lambda(\dot{x}^2 + \dot{y}^2) + l_1\dot{x} + l_2\dot{y} - V \tag{2}$$

where Λ, V, l_1, l_2 are certain functions of x, y.

The Lagrangian system (2) admits the Jacobi integral:

$$I_1 = \frac{1}{2}\Lambda(\dot{x}^2 + \dot{y}^2) + V = h. \tag{3}$$

If this system admits an additional first integral, independent of equation (3), then this system is integrable, i.e. the solution of the mechanical problem reduces to a number of quadratures and to the inversion of certain integrals. This is always guaranteed by the Liouville theorem for the equivalent Hamiltonian system (see, for example, [4]). However, the final solution—the explicit expressions of x, y as functions of t—is not always the principal aim in the mechanical problem. The greater part of the qualitative properties of motion can be deduced directly from the forms of the two known first integrals of the problem.

By transformation to isometric coordinates, the number of functions figuring in the Lagrangian is reduced from six in equation (1) to four in equation (3). Now we make a further step. Let us introduce a new independent variable τ by the relation

$$\mathrm{d}t = \Lambda\,\mathrm{d}\tau \tag{4}$$

[1] Transformation to isometric coordinates on the configurational space.

so that

$$\dot{x} = \frac{1}{\Lambda}x^* \qquad \dot{y} = \frac{1}{\Lambda}y^* \tag{5}$$

where the asterisk denotes differentiation with respect to τ. The Lagrangian (2) takes the form

$$L = \tfrac{1}{2}(x^{*2} + y^{*2}) + l_1 x^* + l_2 y^* + U \tag{6}$$

in which

$$U = \Lambda(h - V). \tag{7}$$

The Jacobi constant h enters as a parameter in the new potential $-U$. To preserve the Jacobi integral (3) under the transformation (4), (5), a superfluous Jacobi constant of the transformed system (6) must be set equal to zero. Thus we have to consider the simultaneous system of differential equations

$$x^{**} + \Omega y^* = \frac{\partial U}{\partial x} \qquad y^{**} - \Omega x^* = \frac{\partial U}{\partial y} \qquad x^{*2} + y^{*2} = 2U \tag{8}$$

where $\Omega = \frac{\partial l_1}{\partial y} - \frac{\partial l_2}{\partial x}$. These equations can be interpreted as describing equienergetic motions of an electrically charged particle under the combined action of electric and magnetic fields. They involve only two functions U and Ω, instead of six in the original system (1). Hence they constitute the simplest basis for the search of the second integral. When $\Omega = 0$ the gyroscopic terms in system (8) vanish and the system becomes time-reversible.

The system (8) enjoys one more useful advantage over Lagrangian and Hamiltonian forms of equations for the same system. It is form-invariant under conformal mappings of the complex plane $z = x + \mathrm{i}y$. It is easy to show that the change of variables

$$z = z(\zeta) \qquad \zeta = \xi + \mathrm{i}\eta \qquad \mathrm{d}\tau = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 \mathrm{d}\tilde{\tau} \tag{9}$$

transforms system (8) to

$$\xi'' + \tilde{\Omega}\eta' = \frac{\partial \tilde{U}}{\partial \xi} \qquad \eta'' - \tilde{\Omega}\xi' = \frac{\partial \tilde{U}}{\partial \eta} \qquad \xi'^2 + \eta'^2 = 2\tilde{U} \tag{10}$$

where $\tilde{U} = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 U$, $\tilde{\Omega} = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 \Omega$ and primes denote differentiation with respect to $\tilde{\tau}$.

Birkhoff raised and completely solved the problem of finding all possible pairs U, Ω for which system (8) admits an additional integral linear in the velocities [9]. He also found that system (8) in the reversible case $\Omega = 0$ admits a quadratic second integral only when it has the structure of a Liouville system [9]. The special case $\Omega = 0$, $\Lambda = 1$ characterizes the problem of motion of a particle under the action of potential forces in the Euclidean plane, which reduce to Liouville's form in elliptic coordinates or any of their degenerations: parabolic, polar and Cartesian coordinates. For detailed analysis and history see [2] and references therein.

In our work [1] a method was developed for the construction of mechanical system (8) for which a second integral exists in the form of a polynomial of arbitrary degree in the velocities. This method, which generalizes that of Birkhoff, has proved effective in constructing several new integrable problems in the dynamics of particles and rigid bodies [1, 2, 10]. It was highly successful in the systematic investigation of the existence of a quadratic second integral in the irreversible case ($\Omega \neq 0$). In [2] all the possible irreversible systems admitting a quadratic integral were classified into three types, according to whether the metric on the configuration space has the structure of a metric in the Euclidean plane, on a surface of revolution or a generic Liouville surface. All cases of the first two types are obtained explicitly. Several many-parameter systems of the third type were also found.

A similar method was developed independently by Hall [13] for the case of plane motion of a particle ($\Lambda = 1$). In view of the complexity of Hall's equations, his method was used only in the reversible case ($\Omega = 0$). In our method, additional transformations of the type (9) are used. This leads to significant simplification in the form of the governing equations and of their solutions.

In this paper, we apply the method introduced in [1], and used also in [2], to construct mechanical systems of the type (10) which admit a cubic second integral, i.e. to find possible function pairs $\tilde{U}$, $\tilde{\Omega}$ and the corresponding coefficients in the integrals. It is worth noting that, after all possible systems of this type are obtained, a general point transformation and suitable time change will lead to all possible systems of the type (1) that have cubic integral.

The general cubic integral can be expressed as the sum of four homogeneous polynomials in ξ', η', all involving ten coefficients as functions of the variables ξ, η. As has been proved in [1] for the general case of a polynomial integral, using a suitable transformation of the form (9) and the Jacobi integral in (10), we can always reduce this integral to the form

$$I = \xi'^3 + P_2\xi'^2 + Q_2\xi'\eta' + P\xi' + Q\eta' + R = \text{const} \tag{11}$$

involving only five functions of ξ, η. Differentiating (11) with respect to τ and using Jacobi's integral again as in [1] and [2], we obtain the system of nonlinear partial differential equations:

$$\begin{aligned}
&\frac{\partial P_2}{\partial \xi} - \frac{\partial Q_2}{\partial \eta} = 0\\
&\frac{\partial P_2}{\partial \eta} + \frac{\partial Q_2}{\partial \xi} - 3\Omega = 0\\
&\frac{\partial P}{\partial \xi} - \frac{\partial Q}{\partial \eta} + 2\Omega Q_2 + 3U_\xi = 0\\
&\frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial \xi} - 2\Omega P_2 = 0\\
&\frac{\partial R}{\partial \xi} + \Omega Q + 2P_2U_\xi + Q_2U_\eta + 2U\frac{\partial Q_2}{\partial \eta} = 0\\
&\frac{\partial R}{\partial \eta} - \Omega P + Q_2U_\xi = 0\\
&PU_\xi + QU_\eta + 2U\frac{\partial Q}{\partial \eta} - 2\Omega U Q_2 = 0.
\end{aligned} \tag{12}$$

That is, seven equations in seven unknown functions. It is not known, however, if this system is integrable, in the sense that its complete set of solutions can be found.

It is evident that any solution of the system (12), involving or not involving arbitrary parameters, can be interpreted as determining a mechanical system that admits a cubic integral on its zero level of the Jacobi integral. If it happens that in this solution the function U has the structure $U + cU_1$, where c is an arbitrary constant, then U_1 can be identified as the coefficient Λ in equation (7) and c as Jacobi's constant. It may even happen, as seen below, that c and U_1 can be chosen in more than one way. In some cases, the same constant enters in one or more of the coefficients of the integral. To obtain an unrestricted integral of motion, this constant should be eliminated in virtue of Jacobi's integral. Note that this process changes the structure of the polynomial second integral and can even change its degree. For some typical such situations see [11].

Regarding the second equation of the last system and the definition of Ω at the beginning of this section, one can now construct a Lagrangian compatible with the integral (11) as

$$L = \tfrac{1}{2}(\xi'^2 + \eta'^2) + \tfrac{1}{3}(P_2\xi' - Q_2\eta') + U. \tag{13}$$

Obviously, this choice is not unique. One can always add the time derivative of an arbitrary function of the two coordinate variables. Note also that a change of the time variable from τ to t according to the relation

$$\mathrm{d}\tau = \chi \mathrm{d}t \tag{14}$$

in which $\chi = \chi(\xi, \eta)$, reduces equation (13) to the form

$$L = \frac{1}{2\chi}(\dot{\xi}^2 + \dot{\eta}^2) + \frac{1}{3}(P_2\dot{\xi} - Q_2\dot{\eta}) + \chi U. \tag{15}$$

This situation is frequently utilized below.

The system of equations of motion (10) is, in general, time irreversible due to the presence of the gyroscopic terms. When $\Omega = 0$, the equations of motion become reversible and the integral of motion must reflect this property. The integral decomposes to even and odd parts, which are also integrals. In our case the integral takes the form

$$I = \xi'^3 + P\xi' + Q\eta' = \text{const} \tag{16}$$

where P, Q satisfy with U the system of three partial differential equations:

$$\frac{\partial P}{\partial \xi} - \frac{\partial Q}{\partial \eta} + 3U_\xi = 0 \qquad \frac{\partial P}{\partial \eta} + \frac{\partial Q}{\partial \xi} = 0 \qquad PU_\xi + QU_\eta + 2U\frac{\partial Q}{\partial \eta} = 0. \tag{17}$$

This system is much simpler than system (12). Any solution of it determines a reversible mechanical system and its cubic integral. Once a solution for this system is obtained, one can try to add a function Ω in a suitable form and corresponding terms to the integral. This procedure has turned out to be more effective in dealing with the system (12). Some examples are given below.

The system (17) can be reduced to a single partial differential equation. In fact, it is always possible to satisfy the first two equations identically by expressing the three unknown functions in the form

$$P = F_{\xi\xi} \qquad Q = -F_{\xi\eta} \qquad U = -\tfrac{1}{3}\nabla^2 F \tag{18}$$

involving only one function. The third equation gives the nonlinear equation of the third order and second degree

$$F_{\xi\xi}\nabla^2 F_\xi - F_{\xi\eta}\nabla^2 F_\eta - 2F_{\xi\eta\eta}\nabla^2 F = 0. \tag{19}$$

However, with further analysis it turns out to be easier to deal with system (17) than with this single equation.

Having no obvious way to obtain the general solution of system (17), one has to add some simplifying assumptions about the structure of the unknown functions. Inspired by the structure of the transformed potential in the integrable case of Goriatchev and Chaplygin (see [1]), we investigate the solution of system (17) for a function U that has the structure

$$U = u(\eta) + v(\eta)\Phi(\xi). \tag{20}$$

A few considerations lead to the following two choices.

2. Construction of reversible systems: first choice

If $\Phi''(\xi) = \text{const} = 2a$ (say), then one can take $\Phi(\xi) = a\xi^2 + b\xi + c$, where b and c are constants. Using this in equations (20) and (17) we come to the conclusion that the remaining functions should have the form

$$P = p + \gamma(a\xi^2 + b\xi + c) \qquad Q = (2a\xi + b)q \qquad U = u + (a\xi^2 + b\xi + c)v \tag{21}$$

where γ is an arbitrary constant and q, v, u, p are functions of η. Equating coefficients of equal powers of ξ, we find that q, v are given by

$$q = \frac{1}{2a}\frac{\mathrm{d}p}{\mathrm{d}\eta} \qquad v = -\frac{1}{3}\gamma - \frac{1}{6a}\frac{\mathrm{d}^2 p}{\mathrm{d}\eta^2} \tag{22}$$

and that u satisfies the equation

$$\frac{\mathrm{d}p}{\mathrm{d}\eta}\frac{\mathrm{d}u}{\mathrm{d}\eta} + 2\frac{\mathrm{d}^2 p}{\mathrm{d}\eta^2}u = -\frac{1}{3}p\left(\frac{\mathrm{d}^2 p}{\mathrm{d}\eta^2} + 2\gamma a\right). \tag{23}$$

Thus it can be expressed as

$$u = \frac{h}{\left(\frac{\mathrm{d}p}{\mathrm{d}\eta}\right)^2} - \frac{1}{3\left(\frac{\mathrm{d}p}{\mathrm{d}\eta}\right)^2}\left[\gamma a p^2 + \int p \frac{\mathrm{d}p}{\mathrm{d}\eta}\frac{\mathrm{d}^2 p}{\mathrm{d}\eta^2}\,\mathrm{d}\eta\right] \tag{24}$$

where h is a new constant. Thus we have expressed all functions in terms of a single function p which can be shown to satisfy the third-order equation

$$\frac{\mathrm{d}p}{\mathrm{d}\eta}\frac{\mathrm{d}^3 p}{\mathrm{d}\eta^3} + 2\left(\frac{\mathrm{d}^2 p}{\mathrm{d}\eta^2}\right)^2 + 2\gamma a\frac{\mathrm{d}^2 p}{\mathrm{d}\eta^2} - 4\gamma^2 a^2 = 0. \tag{25}$$

Here we have separated the integration constant h for its importance in all of what follows. Summing up, we can now construct the restricted integrable Lagrangian

$$\begin{aligned} L = \frac{1}{2}(\xi'^2 + \eta'^2) + \frac{h}{\left(\frac{\mathrm{d}p}{\mathrm{d}\eta}\right)^2} - \frac{1}{3\left(\frac{\mathrm{d}p}{\mathrm{d}\eta}\right)^2}\left[\frac{1}{2}\gamma a p^2 + \int p\frac{\mathrm{d}p}{\mathrm{d}\eta}\frac{\mathrm{d}^2 p}{\mathrm{d}\eta^2}\,\mathrm{d}\eta\right] \\ - \left(\frac{1}{6}\gamma + \frac{1}{6a}\frac{\mathrm{d}^2 p}{\mathrm{d}\eta^2}\right)(a\xi^2 + b\xi + c) \end{aligned} \tag{26}$$

admitting the integral

$$I = \xi'^3 + [p + \gamma(a\xi^2 + b\xi + c)]\xi' + \frac{1}{2a}\frac{\mathrm{d}p}{\mathrm{d}\eta}(2a\xi + b)\eta' \tag{27}$$

where p is any solution of equation (25). For clarity of the results, we distinguish the following cases, in each of which it is useful to follow a different path to the final solution.

2.1. The Holt system

When $\gamma = 0$, equation (25) has the general solution

$$p(\eta) = K(\eta - \eta_0)^{\frac{4}{3}} - 6h_1$$

in which K, η_0, h_1 are arbitrary constants. Without loss of generality we take $\eta_0 = 0$ and rename the parameter K. This leads to the construction of the Lagrangian

$$L = \frac{1}{2}(\xi'^2 + \eta'^2) - \left[\frac{a\left(\frac{3}{4}\eta^2 + \xi^2\right) + b\xi - h_2}{\eta^{\frac{2}{3}}} - h_1\right] \tag{28}$$

and the two integrals of the system

$$\begin{aligned} I_1 &= \frac{1}{2}(\xi'^2 + \eta'^2) + \frac{a\left(\frac{3}{4}\eta^2 + \xi^2\right) + b\xi - h_2}{\eta^{\frac{2}{3}}} - h_1 = 0 \\ I &= \xi'^3 + \left(\frac{27}{2}a\eta^{\frac{4}{3}} - 6h_1\right)\xi' - 9(2a\xi + b)\eta^{\frac{1}{3}}\eta'. \end{aligned} \tag{29}$$

This system characterizes a particle moving in the $\xi\eta$-plane under potential forces involving the three parameters a, b and h_2. The additive parameter h_1 in the Lagrangian (28) does not enter in the equations of motion and can be discarded from the Lagrangian. This constant, however, enters in the energy integral as the arbitrary energy constant and should be substituted from that integral in equations (29) to give the unrestricted second integral of motion

$$I = 2\xi'^3 + 3\xi'\eta'^2 + 3\frac{2(a\xi^2 + b\xi - h_2) - 3a\eta^2}{\eta^{\frac{2}{3}}}\xi' + 9(2a\xi + b)\eta^{\frac{1}{3}}\eta'. \tag{30}$$

This system reduces to Holt's system [17] when one sets $b = 0$ but the case $a = 0, b \neq 0$ is not a subcase of Holt's system.

The presence of four parameters in the Lagrangian (28) makes it possible to generate other Lagrangian systems by means of a time transformation as described in the introduction. As an example we use the transformation

$$\mathrm{d}\tau = \tfrac{2}{3}y\,\mathrm{d}t \qquad \xi = x \qquad \eta = \left(\tfrac{2}{3}y\right)^{\frac{3}{2}} \tag{31}$$

to obtain the integrable Lagrangian

$$L = \frac{1}{2}\left(\frac{3\dot{x}^2}{2y} + \dot{y}^2\right) - \left[a\left(x^2 + \frac{2}{9}y^3\right) + bx - \frac{2}{3}h_1 y\right] + h_2 \tag{32}$$

for which h_2 is the arbitrary energy constant. Note that h_1 is now a parameter in the potential. The cubic integral can be deduced from that in equations (29) by affecting the transformation (31). It may be interesting to note that the line element on the configuration space $\mathrm{d}s^2 = \frac{3\,\mathrm{d}x^2}{2y} + \mathrm{d}y^2$ has negative Gaussian curvature $\kappa = -\frac{3}{4y^2}$.

2.2. The generic case

When $\gamma \neq 0$, one can set, without loss of generality, $\gamma = 1/2$. Introducing the notation

$$p'(\eta) = \sqrt{\zeta} \qquad p''(\eta) = z \tag{33}$$

and thus admitting that

$$\eta = \frac{1}{2}\int \frac{\mathrm{d}\zeta}{z\sqrt{\zeta}} \qquad p = \frac{1}{2}\int \frac{\mathrm{d}\zeta}{z} \tag{34}$$

we reduce equation (25) to the separable first-order equation

$$\frac{\zeta\,\mathrm{d}z}{\mathrm{d}\zeta} + \frac{(2z - a)(z + a)}{2z} = 0. \tag{35}$$

The general solution of equation (35) can be readily obtained in the form

$$\zeta^3\left(z - \frac{a}{2}\right)(z + a)^2 = \text{const}. \tag{36}$$

The problem now arises that neither of the variables z and ζ admits a simple expression in terms of the other, to be effectively used in equation (34). Therefore we try a rational parametrization of the family of curves (36) in the form

$$z = \frac{a}{2}\frac{y^3 - 2b_0}{y^3 + b_0} \qquad \zeta = \frac{y^3 + b_0}{y^2} \tag{37}$$

involving the parameter b_0. Substituting this into equation (36) we obtain, for the constant on the right-hand side, the arbitrary value $-\frac{27}{8}b_0 a^3$. It is more useful to use for the constant b_0 in equation (36) the ratio of two constants $\frac{C}{B}$, and thus we obtain the final parametrization of the family of curves

$$z = \frac{a}{2}\frac{By^3 - 2C}{By^3 + C} \qquad \zeta = \frac{By^3 + C}{By^2}. \tag{38}$$

Inserting the previous expressions into the results (34), (26) and (27), we finish constructing the system with the Lagrangian

$$L = \frac{1}{2}\left[x'^2 + \frac{By^3 + C}{ay^4}y'^2\right] + \frac{hy^2}{a(By^3 + C)} - \frac{By}{4} - \frac{By^3(ax^2 + bx + c)}{4(By^3 + C)} \tag{39}$$

and the conditional integral

$$I = x'^3 + \frac{2By^3 - C + y^2(ax^2 + bx + c)}{2y^2}x' - \frac{(By^3 + C)(2ax + b)}{2ay^3}y' \tag{40}$$

valid on its zero-energy level.

In the expressions (39) and (40) and in the rest of this section the variable ξ has been renamed x. We also use the freedom to change the variable τ to t by a relation of the type (14)

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{G} \tag{41}$$

where $G = G(x, y)$ is a given function. After this, expressions (39) and (40) acquire the forms

$$L = \frac{1}{2}G\left[\dot{x}^2 + \frac{By^3 + C}{ay^4}\dot{y}^2\right] + \frac{1}{G}\left[\frac{hy^2}{a(By^3 + C)} - \frac{By}{4} - \frac{By^3(ax^2 + bx + c)}{4(By^3 + C)}\right] \tag{42}$$

and

$$I = (G\dot{x})^3 + G\frac{2By^3 - C + y^2(ax^2 + bx + c)}{2y^2}\dot{x} - G\frac{(By^3 + C)(2ax + b)}{2ay^3}\dot{y}. \tag{43}$$

Formulae (42) and (43) still characterize a system integrable on its zero-energy level for any reasonable choice of the function G. However, only certain choices allow a suitable additive arbitrary constant to appear in the Lagrangian and thus to be interpreted as the energy constant. To clarify this situation we proceed as follows. Firstly, we rename the parameters

$$h = \frac{h_1}{4} + \alpha a K \qquad b = b_1 - 4\delta K \qquad c = c_1 - 4\beta K$$

and thus introduce the new parameters α, β, δ and K. Secondly, we use a time transformation of the type (41) with

$$G = \frac{y^2(\alpha + \beta y + \delta xy)}{By^3 + C}. \tag{44}$$

We now write down the resulting unconditional Lagrangian system and its two first integrals in their final forms:

$$\begin{aligned} L = {} & \frac{1}{2}(\alpha + \beta y + \delta xy)\left[\frac{y^2\dot{x}^2}{By^3 + C} + \frac{\dot{y}^2}{ay^2}\right] \\ & + \frac{1}{4(\alpha + \beta y + \delta xy)}\left\{\frac{h_1}{Ba} - y(By + ax^2 + b_1 x + c_1) - \frac{C}{y}\right\} + K \end{aligned} \tag{45}$$

$$I_1 = \frac{1}{2}(\alpha + \beta y + \delta x y)\left[\frac{y^2\dot{x}^2}{By^3 + C} + \frac{\dot{y}^2}{ay^2}\right] - \frac{1}{4(\alpha + \beta y + \delta x y)}\left\{\frac{h_1}{Ba} - y(By + ax^2 + b_1 x + c_1) - \frac{C}{y}\right\} = K \tag{46}$$

$$I_2 = \frac{y^6(\alpha + \beta y + \delta x y)^3}{(By^3 + C)^3}\dot{x}^3 + \frac{1}{2}(\alpha + \beta y + \delta x y)\left\{\frac{2By^3 - C + (ax^2 + b_1 x + c_1)y^2}{By^3 + C}\dot{x} - \frac{2ax + b_1}{ay}\dot{y} + \frac{4}{B}I_1\left[-\frac{y^2(\beta + \delta x)}{By^3 + C}\dot{x} + \frac{\delta}{ay}\dot{y}\right]\right\}. \tag{47}$$

Note that the presence of the arbitrary parameter K in equation (45) is insignificant and it can be ignored. The same arbitrary constant K is now interpreted as the value of the energy integral I_1. Note also that I_1 in equation (47) should be replaced by its expression (47) in terms of the state variables.

The system (45)–(47) involves nine arbitrary parameters α, β, δ, B, C, a, h, b_1 and c_1. The first six parameters contribute to the structure of the line element on the configuration manifold as seen from equation (45) and the last three enter only in the potential part of the Lagrangian. We are not able, at present, to give a mechanical interpretation for the full system. The following two special cases may indicate the richness of this system. For these, we provide only L and I_2; the variable y is transformed so that the metric is reduced again to the semi-geodesic form and indices are dropped from the parameters b and c.

1. The first system is obtained from equation (45) by setting $\beta = \delta = 0$ and $\alpha = 1$. It has the Lagrangian

$$L = \frac{1}{2a(B\,\mathrm{e}^{ay} - 2C\,\mathrm{e}^{-2ay})}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{4}aB^2\,\mathrm{e}^{2ay} + \frac{1}{2}aBC\,\mathrm{e}^{-ay} - \frac{1}{4}(ax^2 + bx + c)Ba\,\mathrm{e}^{ay} \tag{48}$$

and admits the unconditional second integral

$$I_2 = \left[\frac{\dot{x}}{a(B\,\mathrm{e}^{ay} - 2C\,\mathrm{e}^{-2ay})}\right]^3 + \frac{2(B\,\mathrm{e}^{ay} + C\,\mathrm{e}^{-2ay}) + ax^2 + bx + c}{2a(B\,\mathrm{e}^{ay} - 2C\,\mathrm{e}^{-2ay})}\dot{x} - \left(x + \frac{b}{2a}\right)\dot{y}. \tag{49}$$

In the special case when $C = 0$, the system (48) and (49) reduces to

$$L = \frac{1}{2Ba}\,\mathrm{e}^{-ay}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{4}aB^2\,\mathrm{e}^{2ay} - \frac{1}{4}(ax^2 + bx + c)Ba\,\mathrm{e}^{ay} + h \tag{50}$$

$$I_2 = \mathrm{e}^{-3ay}\dot{x}^3 + \tfrac{1}{2}B^2a^2[2B + \mathrm{e}^{-ay}(ax^2 + bx + c)]\dot{x} - \tfrac{1}{2}B^3a^2(2ax + b)\dot{y}. \tag{51}$$

It is of interest to note that the line element of the last system has constant negative Gaussian curvature $-\frac{1}{4}a^2$. This system can be interpreted as an integrable case of motion on the pseudo-sphere.

2. The second system corresponds to the case $\alpha = \delta = 0$. The final result for this case is

$$L = \frac{ay^6\dot{x}^2}{8(By^6 - 2C)} + \frac{1}{2}\dot{y}^2 - \frac{1}{16}a^3B\left(By^2 - \frac{2C}{y^4} - \frac{4h}{aBy^2} + ax^2 + bx\right) \tag{52}$$

$$I_2 = \frac{Cy^{12}}{(By^6 - 2C)^3}\dot{x}^3 - \frac{2y^6}{a(By^6 - 2C)}\dot{x}\dot{y}^2 + \frac{ay^2(4BaC + 4hy^2 + aB^2y^6)}{4(By^6 - 2C)}\dot{x} - \frac{1}{2}Ba(2ax + b)y\dot{y}. \tag{53}$$

This system is also new. The Gaussian curvature of the configuration space is

$$\kappa = -\frac{6C(7By^6 + 4C)}{y^2(By^6 - 2C)^2}.$$

When $C = 0$ the last system can be identified with the Holt superintegrable separable system describing the plane motion of a particle [17].

3. Reversible systems: second choice

If $\Phi''(\xi)$ is not identically constant then

$$P = p_0 + p(\eta)\Phi(\xi) \qquad Q = q(\eta)\Phi'(\xi) \tag{54}$$

where p_0 is a constant,

$$\Phi(\xi) = \begin{cases} A\cos\sqrt{\mu}(\xi - \xi_0) & \text{when} \quad \mu > 0 \\ A\,\mathrm{e}^{\sqrt{-\mu}\xi} + B\,\mathrm{e}^{-\sqrt{-\mu}\xi} & \text{when} \quad \mu < 0 \end{cases} \tag{55}$$

and the four functions $p(\eta), q(\eta), u(\eta), v(\eta)$ satisfy the five equations

$$\begin{array}{ll} p'(\eta) - \mu q(\eta) = 0 & q'(\eta) - p(\eta) - 3v(\eta) = 0 \\ 2q'(\eta) + q(\eta)\frac{u'(\eta)}{u(\eta)} + p_0\frac{v(\eta)}{u(\eta)} = 0 & 3q'(\eta) + q(\eta)\frac{v'(\eta)}{v(\eta)} - 3v(\eta) = 0 \end{array} \tag{56}$$

where μ is a separation constant ($\mu \neq 0$). The solution of this system can be written as

$$q = \frac{p'(\eta)}{\mu} \qquad v = \frac{p''(\eta) - \mu p(\eta)}{3\mu} \qquad u = \frac{c}{q^2} - \frac{p_0}{6\mu^2 q^2}[p'^2(\eta) - \mu p^2(\eta)] \tag{57}$$

where c is an integration constant and either $p(\eta)$ satisfies the third-order nonlinear differential equation

$$\frac{\mathrm{d}p}{\mathrm{d}\eta}\frac{\mathrm{d}^3p}{\mathrm{d}\eta^3} + 2\left(\frac{\mathrm{d}^2p}{\mathrm{d}\eta^2}\right)^2 - \mu p\frac{\mathrm{d}^2p}{\mathrm{d}\eta^2} - \mu\left(\frac{\mathrm{d}p}{\mathrm{d}\eta}\right)^2 - \mu^2p^2 = 0. \tag{58}$$

Equation (58) admits reduction to a first-order equation. Introducing new variables z and Z by the relations

$$\frac{p'(\eta)}{p(\eta)} = \sqrt{z} \qquad \sqrt{z}z'(\eta) = 2Z \tag{59}$$

we can write

$$p = \mathrm{e}^{\frac{1}{2}\int\frac{z}{Z}\,\mathrm{d}z} \qquad \eta = \frac{1}{2}\int\frac{\sqrt{z}}{Z}\,\mathrm{d}z \tag{60}$$

and the whole problem is reduced to the Abel differential equation

$$Z\left(2\frac{\mathrm{d}Z}{\mathrm{d}z} + 7z - \mu\right) + z(z-\mu)(3z+\mu) = 0. \tag{61}$$

The latter equation was obtained in our earlier work [1], in the search for integrable motions of a rigid body system with a cubic second integral. The general solution of this equation was found but used only in the interest of the rigid body dynamics. Recently, in the search for integrable systems on S^2 possessing an integral cubic in momenta [16], Selivanova obtained a pair of first-order equations, equivalent (can be reduced) to equation (58). Having not found their solution, she tried a series of existence theorems and asymptotic properties of the solution [16].

Here we construct the general solution of equation (61) using the Darboux method (see, for example, [12]). We first find by trial some simple polynomial and algebraic particular solutions

$$\begin{aligned}
Z_1 &= -z(z-\mu) \\
Z_2 &= -\frac{1}{4}(z-\mu)(3z+\mu) \\
Z_{3,4} &= -\frac{1}{9}(3z+\mu)\left[(3z-\mu) \pm \sqrt{\mu(3z+\mu)}\right] \\
Z_{5,6} &= -(z-\mu)\left[z+\frac{\mu}{2} \pm \sqrt{\frac{\mu}{6}\left(z+\frac{\mu}{2}\right)}\right].
\end{aligned} \tag{62}$$

Moreover, we try to put the solution in the form

$$(Z-Z_1)^{d_1}(Z-Z_2)^{d_2}[(Z-Z_3)(Z-Z_4)]^{d_3}[(Z-Z_5)(Z-Z_6)]^{d_4} = \text{const} \tag{63}$$

where $d_1, \ldots, d_4$ are constants to be determined. Differentiating with respect to z and using equation (61) we find two independent combinations leading to the two equivalent forms of the first integral (the general solution)

$$\frac{[Z+z(z-\mu)]^2\left[\left(Z+z^2-\frac{\mu^2}{9}\right)^2-\frac{\mu}{3}\left(z+\frac{\mu}{3}\right)^2\right]}{\left\{\left[Z+(z-\mu)\left(z+\frac{\mu}{2}\right)\right]^2-\frac{\mu}{6}(z-\mu)^2\left(z+\frac{\mu}{2}\right)\right\}^2} = \text{const} \tag{64}$$

and

$$\frac{[4Z+(z-\mu)(3z+\mu)]^3}{\{3[2Z+(z-\mu)(2z+\mu)]^2-\mu(z-\mu)^2(2z+\mu)\}^2} = \text{const} = \frac{\alpha^3}{\beta^2\mu^2} \tag{65}$$

where α and β are arbitrary constants.

Equations (64) and (65) are of the fourth degree in Z and the eighth degree in z. Instead of trying an explicit solution, it is more convenient to parametrize that curve. We soon realize that the last form of the integral is most suited for this. Introducing a new variable ν by the relations

$$\begin{aligned}
4Z+(z-\mu)(3z+\mu) &= \frac{\alpha(z-\mu)^2}{2\nu^2} \\
3[2Z+(z-\mu)(2z+\mu)]^2-\mu(z-\mu)^2(2z+\mu) &= \frac{\mu\beta(z-\mu)^3}{4\nu^3}
\end{aligned} \tag{66}$$

so that equation (65) is satisfied identically, and solving those relations we obtain

$$\begin{aligned}
z &= \frac{\mu(3\alpha^2+4\nu-12\alpha\nu^2-4\nu^4)}{(2\nu^2+\alpha)^2} \\
Z &= \frac{2\mu^2(\beta-6\alpha\nu-4\nu^3)(\alpha\beta-12\alpha^2\nu-6\beta\nu^2+8\alpha\nu^3)}{9(2\nu^2+\alpha)^4}.
\end{aligned} \tag{67}$$

Substituting into equations (60) and (57) we obtain the full parametric solution of equation (58) and construct the system with the Lagrangian

$$\begin{aligned}
L = \frac{1}{2}\left[\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 + 3\frac{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}{\mu(4\nu^3+6\alpha\nu-\beta)^2}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\tau}\right)^2\right] + \frac{(2p_0\nu+3h_1)(4\nu^3+6\alpha\nu-\beta)}{3(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)} \\
+ \frac{4(4\nu^3+6\alpha\nu-\beta)^{3/2}}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}\Phi(\xi)
\end{aligned} \tag{68}$$

and the second first integral

$$I = \left(\frac{d\xi}{d\tau}\right)^3 + \left[p_0 + \frac{6(2\nu^2+\alpha)\Phi(\xi)}{\sqrt{4\nu^3+6\alpha\nu-\beta}}\right]\frac{d\xi}{d\tau} - \frac{6(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)\Phi'(\xi)}{\mu(4\nu^3+6\alpha\nu-\beta)^{3/2}}\left(\frac{d\nu}{d\tau}\right). \tag{69}$$

This integral is valid only on the zero-energy level of this system. However, the Lagrangian (68) contains two free parameters p_0, h_1 and a change (4) of the independent variable can transform the system to an unconstrained one. This is done in the next section, after we complete the second stage of building the system extending the last one to the irreversible case.

4. Irreversible systems

Now we turn our attention to the case of an irreversible system. The coefficients in the integral of such a system in its normalized form (11) together with the two functions U and Ω satisfy the system of partial differential equations (12). For simplicity we restrict the consideration to the case when $\Omega = \Omega(\eta)$ only, while U still has the same structure as in equation (20), i.e. $U = u(\eta) + v(\eta)\Phi(\xi) + u_1(\eta)$. Inserting those expressions into equation (12) we easily conclude that a possible solution of the simplest structure can be tried in the form

$$\begin{aligned} &P_2 = P_2(\eta) \qquad Q_2 = 0 \\ &P = p_0 + p(\eta)\Phi(\xi) + p_1(\eta) \qquad Q = q(\eta)\Phi'(\xi) \\ &R = R_0(\eta) + R_1(\eta)\Phi(\xi). \end{aligned} \tag{70}$$

The newly introduced functions satisfy the equations

$$\begin{aligned} &3\Omega(\eta) - P_2'(\eta) = 0 \\ &p_1'(\eta) - \tfrac{2}{3}P_2(\eta)P_2'(\eta) = 0 \\ &p_1 v + q u_1'(\eta) + 2u_1(\eta)q'(\eta) = 0 \\ &R_0'(\eta) - \Omega(p_0 + p_1(\eta)) = 0 \\ &R_1(\eta) + \Omega(\eta)q + 2vP_2(\eta) = 0 \\ &R_1'(\eta) - p\Omega(\eta) = 0. \end{aligned} \tag{71}$$

The first five of these equations can be solved in terms of one unknown function P_2

$$\begin{aligned} \Omega(\eta) &= \frac{1}{3}P_2'(\eta) \\ p_1(\eta) &= \frac{1}{3}P_2^2(\eta) \\ u_1(\eta) &= -\frac{1}{3q^2}\int qvP_2^2(\eta)\,d\eta \\ R_0(\eta) &= \frac{1}{3}\left[p_0P_2(\eta) + \frac{1}{9}P_2^3(\eta)\right] \\ R_1(\eta) &= -\left(\frac{1}{3}qP_2'(\eta) + 2vP_2(\eta)\right). \end{aligned} \tag{72}$$

Here we have ignored an integration constant in $p_1(\eta)$ since it can be absorbed in the already existing arbitrary constant p_0. Note that the integration constant in $R_0(\eta)$ is also insignificant, since it adds a constant to the second integral of motion.

Also, from the compatibility of the last equation of (71) with the one before it, we obtain a linear equation determining $P_2(\eta)$ in the form

$$p'(\eta)P_2''(\eta) + [3p''(\eta) - p]P_2'(\eta) + 6v'(\eta)P_2(\eta) = 0. \tag{73}$$

Taking the formulae of subsection 2.1.1 into account we can reduce equation (73) to the more convenient form

$$\frac{\mathrm{d}^2 P_2}{\mathrm{d}\nu^2} - 4\frac{4\nu^3+6\alpha\nu-\beta}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}\frac{\mathrm{d}P_2}{\mathrm{d}\nu} - 4\left[4\frac{(4\nu^3+6\alpha\nu-\beta)^2}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)^2} + 9\frac{2\nu^2+\alpha}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}\right]P_2 = 0 \tag{74}$$

whose general solution can be written as

$$P_2 = 3\frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4} \tag{75}$$

where C and D are arbitrary constants.

Inserting this expression into equation (72) and using equations (70), (13) and (11), we construct the Lagrangian

$$\begin{aligned} L = {} & \frac{1}{2}\left[\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 + 3\frac{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}{\mu(4\nu^3+6\alpha\nu-\beta)^2}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\tau}\right)^2\right] \\ & + \frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}\frac{\mathrm{d}\xi}{\mathrm{d}\tau} + \frac{(4\nu^3+6\alpha\nu-\beta)}{3(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)} \\ & \times\left[2p_0\nu + 3h_1 + \frac{2C^2\nu + 6CD(2\nu^2-\alpha) + 6D^2(4\nu^3-\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}\right] \\ & + \frac{4(4\nu^3+6\alpha\nu-\beta)^{3/2}}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}\Phi(\xi) \end{aligned} \tag{76}$$

and construct the second first integral

$$\begin{aligned} J = {} & \left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^3 + 3\frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 \\ & + \left\{p_0 + \frac{6(2\nu^2+\alpha)\Phi(\xi)}{\sqrt{4\nu^3+6\alpha\nu-\beta}} + 3\left[\frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}\right]^2\right\}\frac{\mathrm{d}\xi}{\mathrm{d}\tau} \\ & - \frac{6(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)\Phi'(\xi)}{\mu(4\nu^3+6\alpha\nu-\beta)^{3/2}}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\tau}\right) \\ & + \frac{4\sqrt{4\nu^3+6\alpha\nu-\beta}[2C\nu+3D(2\nu^2+\alpha)]}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}\Phi(\xi) \\ & + \frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4} \\ & \times\left\{p_0 + \left[\frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}\right]^2\right\} \end{aligned} \tag{77}$$

valid on the zero level of the Jacobi integral

$$J_1 = \frac{1}{2}\left[\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 + 3\frac{(3\alpha^2+4\nu-12\alpha\nu^2-4\nu^4)}{\mu(4\nu^3+6\alpha\nu-1)^2}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\tau}\right)^2\right] - \frac{(4\nu^3+6\alpha\nu-1)}{3(3\alpha^2+4\nu-12\alpha\nu^2-4\nu^4)}$$
$$\times\left[2p_0\nu+3h_1+\frac{2C^2\nu+6CD(2\nu^2-\alpha)+6D^2(4\nu^3-\beta)}{3\alpha^2+4\nu-12\alpha\nu^2-4\nu^4}\right]$$
$$-\frac{4(4\nu^3+6\alpha\nu-1)^{3/2}}{(3\alpha^2+4\nu-12\alpha\nu^2-4\nu^4)}\Phi(\xi) = 0. \qquad (78)$$

Thus we have completed dressing the restricted system (68) and (69) by gyroscopic forces. It remains now to construct the equivalent unrestricted systems.

4.1. The generic unconditional case

We first express the parameters p_0 and h_1 in terms of five new parameters

$$p_0 = \tfrac{3}{2}(a+\gamma K) \qquad h_1 = b+\delta K \qquad (79)$$

and then we perform the time transformation

$$\mathrm{d}\tau = \frac{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}{(\gamma\nu+\delta)(4\nu^3+6\alpha\nu-\beta)}\,\mathrm{d}t \qquad (80)$$

to the above system. Thus we obtain the Lagrangian

$$L = \frac{1}{2}\left[\frac{(\gamma\nu+\delta)(4\nu^3+6\alpha\nu-\beta)}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}\left(\frac{\mathrm{d}\xi}{\mathrm{d}t}\right)^2 + \frac{3(\gamma\nu+\delta)}{\mu(4\nu^3+6\alpha\nu-\beta)}\left(\frac{\mathrm{d}\nu}{\mathrm{d}t}\right)^2\right]$$
$$+\frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}\frac{\mathrm{d}\xi}{\mathrm{d}t}+\frac{1}{3(\gamma\nu+\delta)}\left[3(a\nu+b)\right.$$
$$\left.+\frac{2C^2\nu+6CD(2\nu^2-\alpha)+6D^2(4\nu^3-\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}+4\sqrt{4\nu^3+6\alpha\nu-\beta}\,\Phi(\xi)\right](+K) \qquad (81)$$

where we have ignored K, a free additive parameter which we will accept as Jacobi's constant for this system. Thus, we have the unconditional integral

$$I_1 = \frac{1}{2}\left[\frac{(\gamma\nu+\delta)(4\nu^3+6\alpha\nu-\beta)}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}\left(\frac{\mathrm{d}\xi}{\mathrm{d}t}\right)^2 + \frac{3(\gamma\nu+\delta)}{\mu(4\nu^3+6\alpha\nu-\beta)}\left(\frac{\mathrm{d}\nu}{\mathrm{d}t}\right)^2\right]$$
$$-\frac{1}{3(\gamma\nu+\delta)}\left[3(a\nu+b)+\frac{2C^2\nu+6CD(2\nu^2-\alpha)+6D^2(4\nu^3-\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}\right.$$
$$\left.+4\sqrt{4\nu^3+6\alpha\nu-\beta}\,\Phi(\xi)\right] = K. \qquad (82)$$

The final form of the second integral is

$$I_2 = \left[\frac{(\gamma\nu+\delta)(4\nu^3+6\alpha\nu-\beta)}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}\dot{\xi}\right]^3$$
$$+3\frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)^3}(\gamma\nu+\delta)^2(4\nu^3+6\alpha\nu-\beta)^2\dot{\xi}^2$$
$$+\left\{\frac{3}{2}(a+\gamma I_1)+\frac{6(2\nu^2+\alpha)\Phi(\xi)}{\sqrt{4\nu^3+6\alpha\nu-\beta}}\right.$$

$$+3\left[\frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}\right]^2\Bigg\}\frac{(\gamma\nu+\delta)(4\nu^3+6\alpha\nu-\beta)}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}\dot{\xi}$$

$$-\frac{6(\gamma\nu+\delta)\Phi'(\xi)}{\mu\sqrt{4\nu^3+6\alpha\nu-\beta}}\dot{\nu}+\frac{4\sqrt{4\nu^3+6\alpha\nu-\beta}[2C\nu+3D(2\nu^2+\alpha)]}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)}\Phi(\xi)$$

$$+\frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}$$

$$\times\left\{\frac{3}{2}(a+\gamma I_1)+\left[\frac{C(2\nu^2+\alpha)+D(4\nu^3-6\alpha\nu+2\beta)}{3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4}\right]^2\right\}. \tag{83}$$

4.2. Special cases

4.2.1. An irreversible generalization of the Toda system. The simplest special case of the latter system is obtained if we set $\alpha = \beta = \delta = a = 0, \gamma = 1$. Firstly, we examine the line element of the configuration manifold in that case. From equation (81) we obtain $ds^2 = d\xi^2 - \frac{3}{4\mu\nu^2}\,d\nu^2$. Choosing $\mu = -3$ and introducing the change of variables $\xi = x, \nu = e^{2y}$, we reduce the metric to the Euclidean form and the Lagrangian (81) reduces to the form

$$L = \tfrac{1}{2}(\dot{x}^2+\dot{y}^2) - \tfrac{1}{2}(C\,e^{-4y}+2D\,e^{-2y})\dot{x} - \tfrac{8}{3}\left(A\,e^{y+\sqrt{3}x}+B\,e^{y-\sqrt{3}x}\right)$$
$$+\tfrac{1}{6}C^2\,e^{-8y}+CD\,e^{-6y}+2D^2\,e^{-4y}-b\,e^{-2y} \tag{84}$$

We write the final form of the cubic integral of the Lagrangian (84)

$$I = \dot{x}^3 - 3\dot{x}\dot{y}^2 - 9\left(\tfrac{1}{2}C\,e^{-4y}+D\,e^{-2y}\right)\dot{x}^2 + \tfrac{3}{2}(C\,e^{-4y}+2D\,e^{-2y})\dot{y}^2 + \Big(8A\,e^{y+\sqrt{3}x}$$
$$+8B\,e^{y-\sqrt{3}x}+4C^2\,e^{-8y}+18CD\,e^{-6y}+24D^2\,e^{-4y}-6b\,e^{-2y}\Big)\dot{x}$$
$$-8\sqrt{3}\left(A\,e^{y+\sqrt{3}x}-B\,e^{y-\sqrt{3}x}\right)\dot{y} - C^3\,e^{-12y}-7C^2D\,e^{-10y}-18CD^2\,e^{-8y}$$
$$+(3Cb-16D^3)\,e^{-6y}+6Db\,e^{-4y}-32D\left(A\,e^{-y+\sqrt{3}x}+B\,e^{-y-\sqrt{3}x}\right)$$
$$-8C\left(A\,e^{-3y+\sqrt{3}x}+B\,e^{-3y-\sqrt{3}x}\right). \tag{85}$$

Expressions (84) and (85) characterize an irreversible system generalizing the reversible Toda-like system obtained by Hall [13] by the presence of the two parameters C and D and it reduces to it when $C = D = 0$. This two-parameter generalization invokes a magnetic field $\Omega = 2\left(C\,e^{-4y}+D\,e^{-2y}\right)$ perpendicular to the plane of motion affecting a unit electric charge on the moving particle and adds to the Hall potential the terms $\frac{1}{6}C^2\,e^{-8y}+CD\,e^{-6y}+2D^2\,e^{-4y}$. In the special case, when $A = B = 0$, the system (84) becomes superintegrable since $\dot{x}$ becomes a cyclic integral itself. If, moreover, $C = D = p_0 = 0$, this superintegrable system reduces to that discussed by Leach [15].

It is also worth noting that the system described by equation (84) can be transformed into another form also describing the motion of a particle in the Euclidean plane. In fact, performing in equation (84) the transformation

$$\xi = \theta \qquad \eta = -\ln r \qquad ()' = r^2\dot{()} \tag{86}$$

we obtain a system that describes—in polar coordinates r, θ—the plane motion of a particle whose potential is not periodic in the angle θ. This can be used as a model for a real problem only if the particle is not permitted to make complete revolution about the origin. When $C = D = 0$ the latter system is reduced to that obtained earlier in [14].

4.2.2. The case of elliptic potential. When $\gamma = b = 0$, $\delta = 1$, one can introduce a new variable y by the relation

$$y = \int \frac{\mathrm{d}\nu}{\sqrt{4\nu^3 + 6\alpha\nu - \beta}}. \tag{87}$$

Noting that

$$\nu = \wp(y; -6\alpha, \beta) \qquad \sqrt{4\nu^3 + 6\alpha\nu - \beta} = \wp'(y; -6\alpha, \beta) \tag{88}$$

one can express the Lagrangian (81) and the integral (83) explicitly as elliptic functions in the variable y.

5. Application to dynamics of rigid bodies

Since it is not known yet where most of the new integrable systems constructed above can be applied, it would be useful to demonstrate the richness of the new results by showing how some of them generalize certain known results. The generalization of the Toda system has been presented in the previous section. Here we show that a wide generalization of the most famous integrable case of motion with a cubic integral in rigid body dynamics results as a special case from the system (81).

5.1. Equations of motion of the rigid body

Consider the problem of motion of a rigid body about a fixed point under the action of a combination of conservative potential and gyroscopic forces, described by the Lagrangian

$$L = \tfrac{1}{2}\boldsymbol{\omega}\mathbf{I}\cdot\boldsymbol{\omega} + \mathbf{l}\cdot\boldsymbol{\omega} - V \tag{89}$$

where $\mathbf{I} = \mathrm{diag}(A, B, C)$ is the inertia matrix of the body. Assume that all the forces acting on the body have the Z-axis (say) as a common axis of symmetry. The potential V and the vector $\mathbf{l}$ depend only on the Eulerian angles θ (of nutation) and φ (of proper rotation) through the direction cosines $\gamma = (\gamma_1, \gamma_2, \gamma_3) = (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta)$ of the Z-axis. The equations of motion of a rigid body are usually written in the Euler–Poisson variables $\boldsymbol{\omega}, \boldsymbol{\gamma}$. For the present problem this form, corresponding to equation (89), is

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times (\boldsymbol{\omega} I + \boldsymbol{\mu}) = \boldsymbol{\gamma} \times \frac{\partial V}{\partial \boldsymbol{\gamma}} \qquad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \mathbf{0} \tag{90}$$

where

$$\boldsymbol{\mu} = \frac{\partial}{\partial\boldsymbol{\gamma}}(\mathbf{l}\cdot\boldsymbol{\gamma}) - \left(\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot\mathbf{l}\right)\boldsymbol{\gamma}. \tag{91}$$

For such a system, the angle of precession ψ around the Z-axis is a cyclic variable. Moreover, we restrict our consideration to the case when the body exhibits axial dynamical symmetry $A = B$ and the vector $\mathbf{l}$ lies along the axis of dynamical symmetry, i.e. $\mathbf{l} = (0, 0, l_3)$. The Lagrangian of the system has the form

$$L = \tfrac{1}{2}[A(p^2 + q^2) + Cr^2] + l_3 r - V. \tag{92}$$

It is more convenient for our purpose to write this Lagrangian explicitly in terms of the Eulerian angles as generalized coordinates

$$L' = \tfrac{1}{2}[A(\dot\theta^2 + \sin^2\theta\dot\psi^2) + C(\dot\psi\cos\theta + \dot\varphi)^2] + l_3(\dot\psi\cos\theta + \dot\varphi) - V. \tag{93}$$

The cyclic integral for equation (93) can be written as

$$D\dot\psi + (C\dot\varphi + l_3)\cos\theta = \mathrm{const} = f \tag{94}$$

where $D = A\sin^2\theta + C\cos^2\theta$, that is

$$\dot{\psi} = \frac{f - (C\dot{\varphi} + l_3)\cos\theta}{D}. \tag{95}$$

Ignoring ψ we construct the Routhian

$$R = \frac{1}{2}\left[\dot{\theta}^2 + \frac{C\sin^2\theta\dot{\varphi}^2}{A - (A-C)\cos^2\theta}\right] + \frac{(fC\cos\theta + Al_3\sin^2\theta)\dot{\varphi}}{A[A - (A-C)\cos^2\theta]} - \frac{1}{A}\left\{V + \frac{(f - l_3\cos\theta)^2}{2[A - (A-C)\cos^2\theta]}\right\} \tag{96}$$

or, if we make the substitution $\cos\theta = u$,

$$R = \frac{1}{2}\left[\frac{\dot{u}^2}{1-u^2} + \frac{C(1-u^2)\dot{\varphi}^2}{A-(A-C)u^2}\right] + \frac{(fCu + Al_3(1-u^2))\dot{\varphi}}{A[A-(A-C)u^2]} - \frac{1}{A}\left\{V + \frac{(f-l_3u)^2}{2[A-(A-C)u^2]}\right\}. \tag{97}$$

5.2. *Identification*

First, we must set $\mu = 1$ and identify the variable ξ with the angle φ. Then, equating the coefficients of $\dot{\varphi}^2$ in equation (97) and $\left(\frac{d\xi}{dt}\right)^2$ in equation (81)

$$\frac{C(1-u^2)}{A-(A-C)u^2} = \frac{(\gamma\nu+\delta)(4\nu^3+6\alpha\nu-\beta)}{(3\alpha^2+4\beta\nu-12\alpha\nu^2-4\nu^4)} \tag{98}$$

we can express u in terms of ν. Inserting this expression into the relation

$$\frac{\dot{u}^2}{1-u^2} = \frac{3(\gamma\nu+\delta)}{\mu(4\nu^3+6\alpha\nu-\beta)}\left(\frac{d\nu}{dt}\right)^2 \tag{99}$$

and demanding that it is satisfied identically for arbitrary values of ν, we obtain only two possible combinations of parameters:

(a) $A = 4C, \alpha = -\frac{1}{2}, \beta = 1, \gamma = -\frac{1}{3}, \delta = -\frac{1}{6}$—for this case we have $\nu = \frac{1}{2}(3\cos^2\theta - 1)$;
(b) $A = \frac{4}{3}C, \alpha = 0, \beta = 1, \delta = 0$—for this case we have $\nu = \cos^{\frac{2}{3}}\theta$.

5.2.1. *Case (a).*

In this case, the Lagrangian (81) can be identified with the Routhian (97) if we take the inertia matrix in the form $\mathbf{I} = \mathrm{diag}(4, 4, 1)$, set the cyclic constant $f = 0$ and choose

$$l_3 = k + \frac{c_1}{\gamma_3^2} - \frac{2c_1}{\gamma_3^4} - \frac{c_2}{\gamma_1^2+\gamma_2^2}$$
$$V = a\gamma_1 + \frac{\lambda}{\gamma_3^2} - \frac{c_1(4k+5c_1+12c_2)}{2\gamma_3^4} + \frac{4c_1^2}{\gamma_3^6} - 2\frac{c_1^2}{\gamma_3^8} + \frac{c_2(k-c_1+2c_2)}{\gamma_1^2+\gamma_2^2} - \frac{c_2^2}{2\left(\gamma_1^2+\gamma_2^2\right)^2} \tag{100}$$

where a, k, λ, c_1, c_2 are arbitrary parameters, introduced instead of the original parameters for convenience. The choice (100) characterizes a new integrable problem in the dynamics of a rigid body. It involves two parameters, c_1 and c_2, more than the case found recently by Yehia [19] ($c_1 = c_2 = 0$), three parameters more than both cases of Sretensky [20] ($c_1 = c_2 = \lambda = 0$) and Goriachev (see, for example, [21]) ($c_1 = c_2 = k = 0$) and four parameters more than the classical case due to Goriachev–Chaplygin [22] ($c_1 = c_2 = k = \lambda = 0$).

The potential term $4s\gamma_1$ is the potential of the rigid body whose centre of mass lies in its equatorial plane in a uniform gravity field. The constant term k in the l_3 component of the gyroscopic vector potential is the gyrostatic moment due to a symmetric rotor spinning regularly around the polar axis of the carrier body. The λ-term with quadratic singularity has appeared first in the works of Goriachev [23] (see, for example, [21]) and is common also in certain other problems in rigid body dynamics [18]. The presence of the parameters c_1 and c_2 leads to the simultaneous appearance in both potential and gyroscopic potentials of singular terms, namely, poles of eighth and fourth degrees. This is a new situation in rigid body dynamics and, in general, in integrable systems. The implications of these singularities on the analytical properties of the solution of the equations of motion and on the qualitative properties of motion should be quite interesting. This has not yet been done even for the singular Goriachev potential.

5.2.2. Case (b). In a similar way we can construct the integrable case of rigid body dynamics in which $A = B = \frac{4}{3}$, $C = 1$ and

$$
\begin{aligned}
l_3 &= n + \frac{1}{\left(\gamma_1^2+\gamma_2^2\right)}\left[3n + c_1\gamma_3^{\frac{4}{3}} + c_2\frac{\left(\gamma_3^2+2\right)}{\gamma_3^{\frac{4}{3}}}\right] \\
V &= \frac{(a+b\gamma_1)}{\gamma_3^{\frac{2}{3}}} + \frac{1}{\left(\gamma_1^2+\gamma_2^2\right)^2}\left[\frac{3n^2}{2}\gamma_3^2\left(\gamma_3^2-4\right) - c_1c_2\left(5\gamma_3^2-2\right) - 3nc_1\gamma_3^{\frac{8}{3}}\right. \\
&\qquad \left. - 3nc_2\gamma_3^{\frac{4}{3}}\left(\gamma_3^2+2\right) - \frac{c_1^2\left(7\gamma_3^2-4\right)}{6\gamma_3^{\frac{2}{3}}} - \frac{c_2^2\left(13\gamma_3^4-8\gamma_3^2+4\right)}{2\gamma_3^{\frac{4}{3}}}\right]
\end{aligned}
\tag{101}
$$

where a, b, n, c_1 and c_2 are arbitrary parameters. This is also a new case. It generalizes by the addition of the parameters n, c_1 and c_2 a known integrable case due to Goriachev [24].

6. Conclusion

The method applied here has once more proved effective in the systematic construction of integrable mechanical systems with a polynomial second invariant, which is, in the present work, of the third degree in velocities. We have restored almost all the previously known results as special cases of more general ones and we have constructed several completely new systems with a cubic integral. Our results have the advantage that the configuration manifold is not Euclidean. This widens the scope of applications. For example, from one of the new systems we have obtained, as special cases, new generalizations of the Toda system and of the two known integrable cases with cubic integrals in rigid body dynamics, together with all their previous generalizations by many authors.

To facilitate the solution of the system of partial differential equations, we have had to impose the condition that after the transformation of the Lagrangian and the invariant of the system to the forms (10) and (11) the force function has the structure $U = u(\eta) + v(\eta)\Phi(\xi)$. The three known systems with a cubic invariant that do not appear in our results do not fulfil this condition. These are:

(a) the case of plane motion of a particle found by Fokas [25] (and later by Inozemtsev [26]);
(b) the case of plane motion of a particle due to Calogero [27] and its degenerate versions (see, for example, [3]);
(c) the case of motion of a particle on a sphere, found recently by Gaffet [28].

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